

Entanglement Degree of Parasupersymmetric Coherent States of Harmonic Oscillator

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We study the boson–parafermion entanglement of the parasupersymmetric coherent states of the harmonic oscillator and derive the degree of entanglement in terms of the concurrence. The conditions for obtaining the maximal entanglement is also examined, and it is shown that in the usual supersymmetry situation we can obtain maximally entangled Bell states.

KEY WORDS: entanglement; parasupersymmetry; coherent states.

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1. INTRODUCTION

A fundamental difference between quantum and classical physics is the possible existence of quantum entanglement between distinct systems (Einstein *et al.*, 1935; Schrödinger, 1935). It exhibits the nature of nonlocal correlation between quantum systems, and plays an essential role in various fields of quantum information theory and provides potential resources for communication and information processing (Bennett *et al.*, 1993, 1996; Bennett and Wiesner, 1992). By definition, a pure quantum state of two or more subsystems is said to be entangled if it is not a product of states of each components. A lot of works have been devoted to the preparation and measurement of entangled states (Akhtarshenas, in press; Liu *et al.*, 2004). The entangled orthogonal states receive much attention in the study of quantum entanglement. However the entangled nonorthogonal states also play an important role in the quantum information processing. Bosonic entangled coherent state (Sanders, 1992) and $SU(2)$ and $SU(1, 1)$ coherent states (Wang *et al.*, 2000) are typical examples of nonorthogonal states. Moreover for general bipartite nonorthogonal states some condition have been found for maximal entanglement (Fu *et al.*, 2001; Wang, 2001)

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Supersymmetric (SUSY) quantum mechanics is considered as a simple realization of SUSY algebra involving the fermionic and the bosonic operators (Cooper *et al.*, 1995; Witten, 1981). The formalism of SUSY quantum mechanics has also been extended for parasupersymmetric (PSUSY) quantum mechanics in order to include symmetry between bosons and parafermions of order p ($= 1, 2, \dots$) (Cooper *et al.*, 1995; Durand *et al.*, 1991; Khare, 1992; Rubakov and Spiridonov, 1993).

In this paper, our goal is to investigate the properties of the entanglement degree between bosons and parafermions of the PSUSY coherent states of the harmonic oscillator which have been recently obtained in Fakhri and Bahadori (2000). The bosonic partner of the PSUSY coherent states is expressed in terms of continuous nonorthogonal states. It is shown that these states can be regarded as the states of two logical qubits, so we can easily calculate the concurrence (Wootters, 1998) of the states; an entanglement measure which has widely been accepted as a measure for two qubit states. The condition for obtaining the maximal entanglement is also examined, and it is shown that in the usual supersymmetry situation we can obtain maximally entangled Bell states.

2. PARASUPERSYMMETRIC QUANTUM MECHANICS

In this section we recall the basic features of PSUSY quantum mechanics of order p ($= 1, 2, \dots$). Let us first define parafermi operators b and b^\dagger of order p as which are known to satisfy the PSUSY algebra

$$b^{p+1} = (b^\dagger)^{p+1} = 0, \quad [[b^\dagger, b], b] = -2b, \quad [[b^\dagger, b], b^\dagger] = 2b^\dagger. \quad (1)$$

and

$$b^p b^\dagger + b^{p-1} b^\dagger b + \dots + b^\dagger b^p = \frac{1}{6} p(p+1)(p+2) b^{p-1}. \quad (2)$$

Now by defining

$$J_+ = b^\dagger, \quad J_- = b, \quad J_3 = \frac{1}{2} [b^\dagger, b], \quad (3)$$

it immediately follows from Equation (1) that the operators J_\pm and J_3 satisfy the $SU(2)$ algebra

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm. \quad (4)$$

Let us now choose J_3 as the third component of the spin $\frac{p}{2}$ representation of the $SU(2)$ group with the following explicit form

$$J_3 = \text{diag} \left(\frac{p}{2}, \frac{p}{2} - 1, \dots, -\frac{p}{2} \right). \quad (5)$$

It is now easy to see that the operators b and b^\dagger can be represented by the following $(p + 1) \times (p + 1)$ matrices

$$(b)_{\alpha\beta} = C_\beta \delta_{\alpha,\beta+1}, \quad (b^\dagger)_{\alpha\beta} = C_\alpha \delta_{\alpha+1,\beta}, \tag{6}$$

where

$$C_\beta = \sqrt{\beta(p - \beta + 1)}. \tag{7}$$

Let us now consider the PSUSY harmonic oscillator Hamiltonian as

$$H_{\text{PSUSY}} = \omega \left(a^\dagger a + \frac{1}{2} \right) - \omega J_3. \tag{8}$$

where a and a^\dagger are the bosonic annihilation and creation operators, where satisfy the commutation relation $[a, a^\dagger] = 1$, and J_3 is as given in Equation (5). The first term describes the Hamiltonian of one-dimensional harmonic oscillator and the term $-\omega J_3$ describes the interaction of spin $\frac{p}{2}$ particle with the uniform magnetic field, therefore the whole PSUSY Hamiltonian describes the motion of a spin $\frac{p}{2}$ particle in an oscillator potential and a uniform magnetic field.

It is not difficult to see that the eigenvalue equation for the Hamiltonian of Equation (8) is

$$H_{\text{PSUSY}} |n_b\rangle | \frac{p}{2}, m \rangle = \omega \left(n_b + \frac{1}{2} - m \right) |n_b\rangle | \frac{p}{2}, m \rangle, \tag{9}$$

where $|n_b\rangle$ are orthonormal eigenvectors of $a^\dagger a$ with eigenvalues n_b ($n_b = 0, 1, \dots$) and properties

$$a |n_b\rangle = \sqrt{n_b} |n_b - 1\rangle, \quad a^\dagger |n_b\rangle = \sqrt{n_b + 1} |n_b + 1\rangle, \tag{10}$$

and $|j, m\rangle$ are orthonormal eigenvectors of J_3 with eigenvalues m ($m = -j, -j + 1, \dots, +j$) and properties

$$J_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle. \tag{11}$$

For a fixed spin $j = \frac{p}{2}$, the vectors $|j, m\rangle$ are related to the parafermi Fock states as $|j = \frac{p}{2}, m\rangle = |n_f\rangle$, where $n_f = \frac{p}{2} - m$ denotes number of parafermions. In the boson–parafermion Fock space representation, the eigenvectors of H_{PSUSY} can be written as

$$|\phi_{n,n_f}\rangle = |n - n_f\rangle |n_f\rangle, \quad n = n_b + n_f, \tag{12}$$

which represent a state with $n_b = n - n_f$ boson and n_f parafermion. It is clear that the spectra corresponding to the state $|\phi_{n,n_f}\rangle$ are $(n + 1)$ -fold degenerate (for $n = 0, 1, \dots, p$), and the spectra for $n \geq p$ are $(p + 1)$ -fold degenerate.

3. PSUSY COHERENT STATES

In this section we review PSUSY coherent states which have been obtained in Fakhri and Bahadori (2000). The PSUSY coherent states are defined as eigenvectors of PSUSY annihilation operator A (Fakhri and Bahadori, 2000)

$$A = aI_{p+1} + \frac{(a^\dagger)^{p-1}}{p!}(b^\dagger)^p, \tag{13}$$

The annihilation character of the operator A becomes clear if we choose a suitable superposition of degenerate eigenvectors of H_{PSUSY} and add the requirement $A|\psi_n\rangle = |\psi_{n-1}\rangle$ (Fakhri and Bahadori, 2000). Now the PSUSY coherent states for PSUSY annihilation operator A is defined by

$$A|Z\rangle = z|Z\rangle, \tag{14}$$

where eigenvalue z is an arbitrary complex number. By expanding $|Z\rangle$ in terms of eigenvectors of H_{PSUSY} as

$$|Z\rangle = \sum_{n=n_f}^{\infty} \sum_{n_f=0}^p \beta_{n_f,n} |n - n_f\rangle |n_f\rangle, \tag{15}$$

and taking into account Equation (14), the following solutions are obtained for expansion coefficients (Fakhri and Bahadori, 2000)

$$\begin{aligned} \beta_{0,n} &= -\frac{\sqrt{n!}}{p(n-p)!} z^{n-p} \beta_{p,p} + \frac{z^n}{\sqrt{n!}} \beta_{0,0}, \quad n \geq 0, \\ \beta_{k,n} &= \frac{z^{n-k}}{\sqrt{(n-k)!}} \beta_{k,k}, \quad k = 1, 2, \dots, p, \quad n \geq k + 1. \end{aligned} \tag{16}$$

By requiring the normalization condition $\langle Z|Z\rangle = 1$, and setting

$$\beta_{0,0} = \alpha_0 Q z^{*p}, \quad \beta_{k,k} = \alpha_k Q z^{p-k}, \quad k = 1, 2, \dots, p, \tag{17}$$

where the coefficients α_k ($k = 0, 1, \dots, p$) are real constant and

$$Q(|z|) = \frac{\exp(-|z|^2/2)}{\sqrt{\sum_{n=0}^{p-1} \left(\alpha_{p-n}^2 + \frac{\alpha_p^2}{p^2} \frac{(p!)^2}{(n!)^2 (p-n)!} \right) |z|^{2n} + \left(\alpha_0 - \frac{\alpha_p}{p} \right)^2 |z|^{2p}}}, \tag{18}$$

the following form have been obtained for PSUSY coherent states of harmonic oscillator (Fakhri and Bahadori, 2000)

$$|Z\rangle = Q \left[\left(\alpha_0 (z^*)^p |z\rangle - \frac{\alpha_p}{p} |z^{(p)}\rangle \right) |0\rangle + |z\rangle \left(\sum_{n_f=1}^p \alpha_{n_f} (z)^{p-n_f} |n_f\rangle \right) \right]. \tag{19}$$

In Equation (19) $|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$ is the nonnormalized ordinary coherent state of the harmonic oscillator and $|z^{(p)}\rangle = \frac{\partial^p}{\partial z^p} |z\rangle$, and the relations

$$\begin{aligned} \langle z|z\rangle &= \exp(|z|^2), \\ \langle z|z^{(p)}\rangle &= z^{*p} \exp(|z|^2), \\ \langle z^{(p)}|z^{(p)}\rangle &= \sum_{n=0}^p \frac{(p!)^2}{(n!)^2(p-n)!} |z|^{2n} \exp(|z|^2), \end{aligned} \tag{20}$$

are also satisfied.

4. DEGREE OF ENTANGLEMENT

From the various measures proposed to quantify entanglement, the entanglement of formation has a special position which in fact intends to quantify the resources needed to create a given entangled state (Bennett *et al.*, 1996). Remarkably, Wootters has shown that the entanglement of formation of a two qubit mixed state ρ is related to a quantity called concurrence as (Wootters, 1998)

$$E_f(\rho) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - C^2}\right), \tag{21}$$

where $H(x) = -x \ln x - (1 - x) \ln (1 - x)$ is the binary entropy and the concurrence $C(\rho)$ is defined by

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{22}$$

where the λ_i are the non-negative eigenvalues, in decreasing order, of the Hermitian matrix $R \equiv \sqrt{\sqrt{\rho} \tilde{\rho} \sqrt{\rho}}$ and

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y), \tag{23}$$

where ρ^* is the complex conjugate of ρ when it is expressed in a standard basis such as $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and σ_y represents the Pauli matrix in local basis $\{|0\rangle, |1\rangle\}$. Furthermore, the entanglement of formation is monotonically increasing function of the concurrence $C(\rho)$, so one can use concurrence directly as a measure of entanglement. For pure state $|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$, the concurrence takes the form

$$C(\psi) = |\langle \psi | \tilde{\psi} \rangle| = 2 |a_{00}a_{11} - a_{01}a_{10}|. \tag{24}$$

In the following we will use the concurrence to quantify the entanglement of the PSUSY coherent states (19). Recall that the state (19) may be written as $|\mu\rangle|u\rangle + |\nu\rangle|v\rangle$ where $\{|\mu\rangle, |\nu\rangle\}$ are in general two nonorthogonal vectors in bosonic space and $\{|u\rangle, |v\rangle\}$ are two orthogonal (but not normalized) vectors in parafermion space. The two nonorthogonal vectors $|\mu\rangle$ and $|\nu\rangle$ are assumed to be linearly independent and span the two-dimensional subspace of the bosonic Hilbert space. Therefore we may readily obtain the concurrence for state (19) by introducing

an orthonormal basis in the subspace spanned by $\{|\mu\rangle, |v\rangle\}$. This can be easily achieved by introducing basis

$$|0\rangle_b = \exp\left(-\frac{|z|^2}{2}\right) \frac{((z^*)^p |z\rangle - |z^{(p)}\rangle)}{\sqrt{\sum_{n=0}^{p-1} \frac{(p!)^2}{(n!)^2 (p-n)!} |z|^{2n}}}, \quad |1\rangle_b = \exp\left(-\frac{|z|^2}{2}\right) |z\rangle, \quad (25)$$

in boson space and

$$|0\rangle_f = |0\rangle, \quad |1\rangle_f = \frac{\sum_{k=1}^p \alpha_k(z)^{p-k} |k\rangle}{\sqrt{\sum_{k=1}^p \alpha_k^2 |z|^{2(p-k)}}}. \quad (26)$$

in parafermion space. Under these basis the entangled PSUSY coherent state $|Z\rangle$ can be considered as a state of two logical qubits with the following form

$$|Z\rangle = a_{00}|0\rangle_b|0\rangle_f + a_{01}|0\rangle_b|1\rangle_f + a_{10}|1\rangle_b|0\rangle_f + a_{11}|1\rangle_b|1\rangle_f, \quad (27)$$

where

$$\begin{aligned} a_{00} &= Q \sqrt{\sum_{n=0}^{p-1} \frac{\alpha_p^2}{p^2} \frac{(p!)^2}{(n!)^2 (p-n)!} |z|^{2n}} \exp(|z|^2/2), \\ a_{01} &= 0, \\ a_{10} &= Q (z^*)^p \left(\alpha_0 - \frac{\alpha_p}{p}\right) \exp(|z|^2/2), \\ a_{11} &= Q \sqrt{\sum_{n=0}^{p-1} \alpha_{p-n}^2 |z|^{2n}} \exp(|z|^2/2). \end{aligned} \quad (28)$$

Equation (24) can be now easily used to calculate the concurrence of PSUSY coherent state of order p as

$$C(p, z) = 2 \frac{\left[\left(\sum_{n=0}^{p-1} \alpha_{p-n}^2 |z|^{2n} \right) \left(\sum_{n=0}^{p-1} \frac{\alpha_p^2}{p^2} \frac{(p!)^2}{(n!)^2 (p-n)!} |z|^{2n} \right) \right]^{1/2}}{\left[\sum_{n=0}^{p-1} \left(\alpha_{p-n}^2 + \frac{\alpha_p^2}{p^2} \frac{(p!)^2}{(n!)^2 (p-n)!} \right) |z|^{2n} + \left(\alpha_0 - \frac{\alpha_p}{p} \right)^2 |z|^{2p} \right]}. \quad (29)$$

By defining

$$\begin{aligned} \mathcal{A} &= \sqrt{\sum_{n=0}^{p-1} \alpha_{p-n}^2 |z|^{2n}}, \\ \mathcal{B} &= \sqrt{\sum_{n=0}^{p-1} \frac{\alpha_p^2}{p^2} \frac{(p!)^2}{(n!)^2 (p-n)!} |z|^{2n}}, \end{aligned} \quad (30)$$

we get the following form for concurrence (29)

$$C(p, z) = \frac{2\mathcal{A}\mathcal{B}}{\mathcal{A}^2 + \mathcal{B}^2 + \left(\alpha_0 - \frac{\alpha_p}{p}\right)^2 |z|^{2p}}. \quad (31)$$

In the following our goal is to investigate the properties of the concurrence given in Equation (29) or (31). First, we remark that state (19) is disentangled, i.e.

$C(p, z) = 0$, if and only if $\alpha_p = 0$. In this particular case we have the following product state

$$|Z\rangle = \frac{|z\rangle \left(\sum_{n_f=0}^{p-1} \alpha_{n_f}(z)^{p-n_f} |n_f\rangle \right)}{\exp(|z|^2/2) \left(\sum_{n_f=0}^{p-1} \alpha_{n_f}^2 |z|^{2(p-n_f)} \right)}. \tag{32}$$

Now, we try to find the situations that the concurrence becomes maximal. It is clear that since \mathcal{A} and \mathcal{B} are independent of α_0 , therefore the first step to maximize $C(p, z)$ is to set $\alpha_0 = \frac{\alpha_p}{p}$, and the problem of maximizing concurrence reduces to the problem of minimizing $1 - C^2(p, z)$ given by

$$1 - C^2(p, z) = \frac{(\mathcal{A}^2 - \mathcal{B}^2)^2}{(\mathcal{A}^2 + \mathcal{B}^2)^2} = \left\{ \frac{\alpha_p^2 \left(\frac{p!}{p^2} - 1 \right) + \sum_{n=1}^{p-1} \left(\frac{\alpha_p^2}{p^2} \frac{(p!)^2}{(n!)^2 (p-n)!} - \alpha_{p-n}^2 \right) |z|^{2n}}{\alpha_p^2 \left(\frac{p!}{p^2} + 1 \right) + \sum_{n=1}^{p-1} \left(\frac{\alpha_p^2}{p^2} \frac{(p!)^2}{(n!)^2 (p-n)!} + \alpha_{p-n}^2 \right) |z|^{2n}} \right\}^2. \tag{33}$$

From Equation (33) it is obvious that if we want to have a maximal entangled state for all eigenvalues z , then the only solution of this equation is obtained for usual SUSY coherent states, i.e. $p = 1$ and $\alpha_0 = \alpha_p$. In this case the maximal entangled SUSY coherent state is the Bell state

$$|Z\rangle = \frac{1}{\sqrt{2}}(|\mathbf{0}\rangle_b |\mathbf{0}\rangle_f + |\mathbf{1}\rangle_b |\mathbf{1}\rangle_f) = \frac{\exp(-|z|^2/2)}{\sqrt{2}} \{ (z^*|z\rangle - |z^{(p)}\rangle) |0\rangle + |z\rangle |1\rangle \}. \tag{34}$$

On the other hand for $p > 1$ there is no solution for the constant coefficients α_k and all z , in which the system exactly reaches to a maximally entangled state such that the concurrence is 1. But for $|z| > 1$ we can find the solutions that we can nearly obtain the maximally entangled state. At this point let us choose the coefficients α_k as

$$\alpha_k = \frac{p!}{p(p-k)! \sqrt{k!}} \alpha_p, \quad k = 1, 2, \dots, p-1. \tag{35}$$

In this case the series in the numerator of Equation (33) vanishes and we obtain

$$C(p, z) = \sqrt{1 - \frac{\left(\frac{p!}{p^2} - 1 \right)^2}{\left(\left(\frac{p!}{p^2} + 1 \right) + 2 \sum_{n=1}^{p-1} \frac{(p!)^2}{p^2 (n!)^2 (p-n)!} |z|^{2n} \right)^2}}. \tag{36}$$

Figure 1 demonstrates the concurrence (36) as a function of p and z . It should be stressed that, although the figure is plotted for continues values of PSUSY parameter p , but only the integer values $p = 1, 2, \dots$ are physically relevant. It

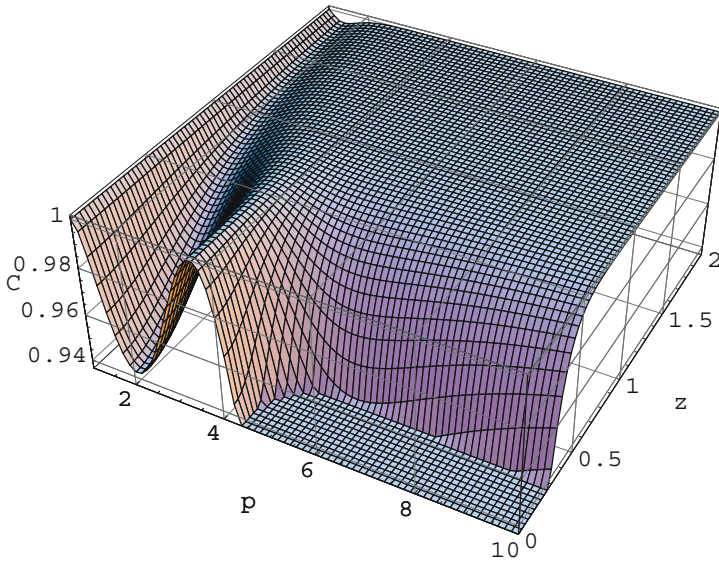


Fig. 1. Concurrence $C(p, z)$ is plotted as a function of p and z . Only integer values of p are physically meaning.

shows that in all cases by increasing the eigenvalue z , the concurrence $C(p, z)$ rapidly reaches to maximum value 1. Indeed we find that for $|z| > 1$ the difference between the maximum value of the concurrence, i.e. $C = 1$, and the concurrence of the maximally entangled state is of the order of less than 10^{-3} .

It is interesting to note that we may yet obtain, exactly, maximal entangled states if we choose α_k such that some of them be dependent to z . In this case one particular set of solutions of the Equation (33) can be obtained if we choose all but one of the coefficients α_k constant, i.e.

$$\begin{aligned} \alpha_k &= \frac{p!}{p(p-k)!\sqrt{k!}}\alpha_p, & k = 1, 2, \dots, p-1, & \quad k \neq p-m, \\ \alpha_{p-m}^2 |z|^{2m} &= \alpha_p^2 \left(\left(\frac{p!}{p^2} - 1 \right) + \frac{(p!)^2}{p^2(m!)^2(p-m)!} |z|^{2m} \right). \end{aligned} \tag{37}$$

Clearly in this case, which is not the only case, we obtain maximum value 1 for concurrence.

5. CONCLUSION

We have studied boson–parafermion entanglement of the parasupersymmetric coherent states of the harmonic oscillator. The concurrence of the state is obtained by using orthonormal basis of both bosonic and parafermionic partner of the states. The condition for obtaining the maximal entanglement is also examined,

and it is shown that in the usual supersymmetry situation we can obtain maximally entangled Bell states. For a general PSUSY coherent state, it is shown that we can approximately obtain the maximal entangled state whenever the value of z is large enough.

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